

A Characterisation of the Euclidean Fourier transform on the Schwartz space

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ABSTRACT. We obtain a characterisation of the Fourier transform on the space of Schwartz class functions on \mathbb{R}^n . The result states that any appropriately additive bijection of the Schwartz space onto itself, which interchanges convolution and pointwise products is essentially the Fourier transform.

1. Introduction

The Fourier transform on various locally compact groups, and its properties with respect to different operations on function spaces on these groups are well understood. The interaction of the Fourier transform with the translations on the groups, and with certain products on the functions defined on these groups have been used to obtain characterisations of the Fourier transform. For more details, refer to [1]-[8], and the references therein.

We denote by $\mathcal{S}(\mathbb{R}^n)$, the Schwartz class of rapidly decreasing functions on \mathbb{R}^n , defined as follows:

For a function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, let

$$\|f\|_{\alpha,\beta} := \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta f(x)|,$$

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where for multi-indices $\alpha = (\alpha_1, \dots, \alpha_n)$, and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we denote $x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}$, and $\partial^\beta = \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \frac{\partial^{\beta_2}}{\partial x_2^{\beta_2}} \dots \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}}$.

The *Schwartz class* of functions, denoted $\mathcal{S}(\mathbb{R}^n)$, or simply \mathcal{S} , is defined to be

$$\mathcal{S}(\mathbb{R}^n) := \{f : \mathbb{R}^n \rightarrow \mathbb{C} : f \in \mathcal{C}^\infty(\mathbb{R}^n), \|f\|_{\alpha, \beta} < \infty \text{ for all } \alpha, \beta \in \mathbb{N}^n\}.$$

Then the space $\mathcal{C}_c^\infty(\mathbb{R}^n)$, also denoted \mathcal{C}_c^∞ , of compactly supported smooth functions defined on \mathbb{R}^n , is a subspace of $\mathcal{S}(\mathbb{R}^n)$.

The topology generated by the family of seminorms $\{\|\cdot\|_{\alpha, \beta} : \alpha, \beta \in \mathbb{N}^n\}$ makes $\mathcal{S}(\mathbb{R}^n)$ into a Fréchet space over the complex numbers. Also, $\mathcal{S}(\mathbb{R}^n)$ is closed under the operations of pointwise and convolution product, where the convolution of functions in $\mathcal{S}(\mathbb{R}^n)$ is defined as

$$(f * g)(x) = \int_{\mathbb{R}^n} f(x - y) g(y) dy, \quad x \in \mathbb{R}^n.$$

For a function $f \in \mathcal{S}(\mathbb{R}^n)$, its Fourier transform $\mathcal{F}f$ is defined as

$$\mathcal{F}f(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.$$

The space of all continuous linear functionals on $\mathcal{S}(\mathbb{R}^n)$ is called the *space of tempered distributions*, and is denoted by $\mathcal{S}'(\mathbb{R}^n)$. We denote the action of $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ on a function $f \in \mathcal{S}(\mathbb{R}^n)$ as $\langle \varphi, f \rangle$.

The operations of pointwise multiplication and convolution of functions in $\mathcal{S}(\mathbb{R}^n)$ can be appropriately extended to $\mathcal{S}'(\mathbb{R}^n)$ as follows: For $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}'(\mathbb{R}^n)$,

$$\begin{aligned} \langle f \cdot \varphi, g \rangle &= \langle \varphi, f \cdot g \rangle \\ \langle f * \varphi, g \rangle &= \langle \varphi, \tilde{f} * g \rangle, \end{aligned}$$

where $\tilde{f}(x) = f(-x)$ for $x \in \mathbb{R}^n$. Then for $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, we have $f \cdot \varphi \in \mathcal{S}'(\mathbb{R}^n)$ and $f * \varphi \in \mathcal{S}'(\mathbb{R}^n)$.

The Fourier transform, initially defined on $\mathcal{S}(\mathbb{R}^n)$, can be extended to the space $\mathcal{S}'(\mathbb{R}^n)$ via

$$\langle \mathcal{F}\varphi, f \rangle = \langle \varphi, \mathcal{F}f \rangle, \text{ for } f \in \mathcal{S}(\mathbb{R}^n), \varphi \in \mathcal{S}'(\mathbb{R}^n).$$

The Fourier transform is a topological isomorphism of $\mathcal{S}'(\mathbb{R}^n)$ onto itself and satisfies

$$\begin{aligned}\mathcal{F}(f \cdot \varphi) &= \mathcal{F}(f) * \mathcal{F}(\varphi) \\ \mathcal{F}(f * \varphi) &= \mathcal{F}(f) \cdot \mathcal{F}(\varphi).\end{aligned}$$

In [2] S. Alesker, S. Artstein-Avidan and V. Milman gave a very interesting characterisation of the Fourier transform on the Schwartz class of functions on \mathbb{R}^n . The precise statement of their result is as follows:

THEOREM 1.1. *Assume that $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ is a bijection which admits a bijective extension $T' : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^n)$ such that for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, we have*

$$T(f * \varphi) = T(f)T(\varphi) \quad \text{and} \quad T(f \cdot \varphi) = T(f) * T(\varphi).$$

Then T is essentially the Fourier transform: that is, for some matrix $B \in GL(n, \mathbb{R})$ with $|\det B| = 1$, we have either $Tf = \mathcal{F}(f \circ B)$ or $Tf = \mathcal{F}(\overline{f \circ B})$ for all functions $f \in \mathcal{S}(\mathbb{R}^n)$.

As the authors of the above result had remarked, the hypotheses of this result involves only *algebraic* properties of the map on the class of tempered distributions, whereas the conclusion states that the map is essentially the Fourier transform.

Motivated by the above result, a characterisation of the Fourier transform on the Schwartz space of the Heisenberg group was obtained in [8]. This result did not involve any hypothesis in terms of the tempered distributions. The anonymous referee of [8] suggested if a characterisation of the Fourier transform on \mathbb{R}^n , without any assumptions on the tempered distributions, could be obtained. This paper is an attempt towards a positive answer to this question.

For a function $f : \mathbb{R}^n \rightarrow \mathbb{C}$, the support of f , denoted $Supp f$, is defined as

$$Supp f := \text{Closure}(\{x \in \mathbb{R}^n : f(x) \neq 0\}).$$

2. A Characterisation of Fourier transform on $\mathcal{S}(\mathbb{R}^n)$

We remark that our results are very much influenced by the those of Alesker et al.[2] and their interesting proofs.

Our main result is the following:

THEOREM 2.1. *Let $T : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ be a bijection satisfying the following conditions for all functions $f, g \in \mathcal{S}(\mathbb{R}^n)$:*

- (a): $T(f + \bar{g}) = T(f) + [T(g)]^*$, where $[Tg]^*(x) = \overline{Tg(-x)}$, $x \in \mathbb{R}^n$.
- (b): $T(f \cdot g) = T(f) * T(g)$,
- (c): $T(f * g) = T(g) \cdot T(f)$.

Then there exists a matrix $B \in GL(n, \mathbb{R})$, with $|\det B| = 1$ such that either $Tf = \mathcal{F}(f \circ B)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, or $Tf = \mathcal{F}(\overline{f \circ B})$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. For $f \in \mathcal{S}(\mathbb{R}^n)$, we have $Tf \in \mathcal{S}(\mathbb{R}^n)$. Since the Fourier transform \mathcal{F} is a bijection on $\mathcal{S}(\mathbb{R}^n)$, there exists unique $g \in \mathcal{S}(\mathbb{R}^n)$ with $Tf = \mathcal{F}g$. Define a map $U : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ as $Uf := g$ if $Tf = \mathcal{F}g$. Then $Tf = \mathcal{F}(Uf)$ for all $f \in \mathcal{S}$. The map U is a bijection of \mathcal{S} onto itself and satisfies the following conditions for all functions $f, g \in \mathcal{S}(\mathbb{R}^n)$:

- (1) $U(f + \bar{g}) = U(f) + \overline{U(g)}$,
- (2) $U(f \cdot g) = U(f) \cdot U(g)$,
- (3) $U(f * g) = U(f) * U(g)$.

The theorem is a then a consequence of the following result, which gives a precise description of the map U .

□

THEOREM 2.2. *Let $U : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ be a bijection satisfying the following conditions for all functions $f, g \in \mathcal{S}(\mathbb{R}^n)$:*

- (1) $U(f + \bar{g}) = U(f) + \overline{U(g)}$,
- (2) $U(f \cdot g) = U(f) \cdot U(g)$,
- (3) $U(f * g) = U(f) * U(g)$.

Then there exists a matrix $B \in GL(n, \mathbb{R})$ with $|\det B| = 1$ such that either $Uf(x) = f(Bx)$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, or $Uf(x) = \overline{f(Bx)}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$.

PROOF. We prove the result in 12 steps.

For $x_0 \in \mathbb{R}^n$, define

$$C(x_0) := \{f \in \mathcal{S}(\mathbb{R}^n) : x_0 \in \text{Supp } f\}.$$

Step 1. Let $f, g \in \mathcal{S}$. If $g = 1$ on $\text{Supp } f$, then $Ug = 1$ on $\text{Supp } Uf$.

Proof of Step 1. Since $g = 1$ on $\text{Supp } f$, we have $f \cdot g = f$. This gives $Uf = U(f \cdot g) = Uf \cdot Ug$, and so $Ug = 1$ on the set $\{x : Uf(x) \neq 0\}$.

Let $x \in \text{Supp } Uf$ with $Uf(x) = 0$. Then there is a sequence $\{x_k\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^n$ with $Uf(x_k) \neq 0$ for all k and $x_k \rightarrow x$ as $k \rightarrow \infty$. Since $Uf(x_k) \neq 0$, we have $Ug(x_k) = 1$ for all k . Hence $Ug(x) = \lim_{k \rightarrow \infty} Ug(x_k) = 1$. Thus $Ug = 1$ on $\text{Supp } Uf$.

Step 2. If $f \in \mathcal{C}_c^\infty$, then $Uf \in \mathcal{C}_c^\infty$.

Proof of Step 2. Choose $f \in \mathcal{C}_c^\infty$ such that $f(x_0) \neq 0$. Choose $g \in \mathcal{S}$ such that $g = 1$ on $\text{Supp } f$. By Step 1, $Ug = 1$ on $\text{Supp } Uf$. Since $Ug \in \mathcal{S}$, we get $\text{Supp } f$ is compact.

Step 3. For any $x_0 \in \mathbb{R}^n$, there exists $y_0 \in \mathbb{R}^n$ such that $Uf \in C(y_0)$ whenever $f \in C(x_0)$.

Proof of Step 3. Let $E := \{f \in \mathcal{S} : f(x_0) \neq 0\}$. Fix a function $g \in \mathcal{C}_c^\infty$ with $g(x_0) \neq 0$. By Step 2, we have $K := \text{Supp } Ug$ is compact.

For $f \in E$, define $K_f := K \cap \text{Supp } Uf$. For functions $f_0 := g, f_1, \dots, f_k \in E$, we have $\prod_{j=0}^k f_j \neq 0$, and so $\prod_{j=0}^k Uf_j = U(\prod_{j=0}^k f_j) \neq 0$, which gives $\bigcap_{j=0}^k K_{f_j} \neq \emptyset$. This means, the collection $\{K_f : f \in E\}$ of closed subsets of K has finite intersection property. Since K is compact, this gives $\bigcap_{f \in E} K_f \neq \emptyset$. Let $y_0 \in \bigcap_{f \in E} K_f$.

Claim. If $f \in C(x_0)$, then $Uf \in C(y_0)$.

Proof of Claim. We prove the claim in two separate cases.

Case 1. $f(x_0) \neq 0$.

Then f never vanishes on a neighborhood, say, V of x_0 . Let $g \in \mathcal{S}$ be such that $f \cdot g = 1$ on V . Choose $h \in \mathcal{S}$ such that $h = 1$ on a neighborhood W of x_0 , and satisfies $W \subseteq \text{Supp } h \subseteq V$. Since $f \cdot g = 1$ on $\text{Supp } h$, by Step 1, we get $U(f \cdot g) = Uf \cdot Ug = 1$ on $\text{Supp } Uh$. Since $h \in E$, by definition, $y_0 \in \text{Supp } Uh$. This implies $Uf(y_0) \neq 0$, and hence $Uf \in C(y_0)$.

Note that all our arguments till now can be applied to the map U^{-1} as well, and so we have proved that $f(x_0) \neq 0$ if and only if $Uf(y_0) \neq 0$.

A function $f \in \mathcal{S}$ is said to satisfy the condition (\star) if the following holds:

$$(\star) \quad f(x_0) = 0 \text{ if and only if } Uf(y_0) = 0.$$

By the above discussion, we have that all functions in \mathcal{S} satisfy condition (\star) .

Case 2. $f(x_0) = 0$.

Suppose $Uf \notin C(y_0)$. Then there is a neighbourhood, say W , of y_0 such that Uf vanishes identically on W . Let $h \in \mathcal{S}$ with $\text{Supp } h \subseteq W$, and $h(y_0) \neq 0$. There exists unique function $g \in \mathcal{S}$ with $Ug = h$. Then $U(f \cdot g) = Uf \cdot Ug = Uf \cdot h \equiv 0$. This gives $f \cdot g \equiv 0$.

On the other hand, since $Ug(y_0) = h(y_0) \neq 0$, by Condition (\star) , we have $g(x_0) \neq 0$, and so g is never zero near x_0 . Since $x_0 \in \text{Supp } f$, this implies $f \cdot g \not\equiv 0$, a contradiction. Thus $Uf \in C(y_0)$.

Step 4. Define a map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ as follows: $Ax = y$ if $Uf \in C(y)$ whenever $f \in C(x)$. Then the map A is well-defined.

Proof of Step 4. Suppose for some $x_0 \in \mathbb{R}^n$, we have $Ax_0 = y_1$ and $Ax_0 = y_2$ with $y_1 \neq y_2$. Let V_1 and V_2 be disjoint neighborhoods of y_1 and y_2 , respectively. There exists functions g_1 and g_2 in \mathcal{S} which are supported in V_1 and V_2 , respectively, such that $g_1(y_1) \neq 0$ and $g_2(y_2) \neq 0$. Let $f_1, f_2 \in \mathcal{S}$ with $Uf_1 = g_1$ and $Uf_2 = g_2$. Then $0 \equiv g_1 \cdot g_2 = Uf_1 \cdot Uf_2 = U(f_1 \cdot f_2)$ and so $f_1 \cdot f_2 \equiv 0$.

On the other hand, as $g(y_j) = Uf_j(y_j) \neq 0$ for $j = 1, 2$, we have by Condition (\star) that $f_j(x_0) \neq 0$ for $j = 1, 2$, which is in contradiction to $f_1 \cdot f_2 \equiv 0$.

Step 5. $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a bijection.

Proof of Step 5. The hypotheses of the theorem hold good for the map U^{-1} as well. Applying the preceding steps to the map U^{-1} gives rise to a well-defined function, say, $B : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Then $B = A^{-1}$, proving that A is bijective.

Our observations can be summarised as:

$$A(\text{Supp } f) = \text{Supp } Uf, \quad \text{for all } f \in \mathcal{S}.$$

Step 6. The map A is a homeomorphism of \mathbb{R}^n onto itself.

Proof of Step 6. Suppose not. Then there exist $x \in \mathbb{R}^n$, and sequence $\{x_k\}$ in \mathbb{R}^n with $x_k \rightarrow x$ as $k \rightarrow \infty$, but Ax_k does not converge to Ax .

Let V be a neighborhood of Ax such that $Ax_k \notin V$ for any k . Let $h \in \mathcal{S}$ with $\text{Supp } h \subseteq V$, and $h(Ax) = 1$. Let $g \in \mathcal{S}$ be such that $Ug = h$. Then $Ax_k \notin \text{Supp } Ug$ for any k , and so $x_k \notin \text{Supp } g$ for any k . This gives $g(x_k) = 0$ for all k , implying $g(x) = 0$, which is not possible by Condition (\star) , since $Ug(Ax) = 1$.

We observe that the above argument holds good when the maps U and A are replaced with U^{-1} and A^{-1} respectively, yielding that $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a homeomorphism.

Step 7. The map A satisfies $A(x + y) = Ax + Ay$ for all $x, y \in \mathbb{R}^n$.

Proof of Step 7. Suppose $A(x + y) \neq Ax + Ay$ for some $x, y \in \mathbb{R}^n$.

Then there exist disjoint neighborhoods V_{x+y}, V_{xy} with $A(x+y) \in V_{x+y}$ and $Ax + Ay \in V_{xy}$. By continuity of the map A , this gives rise to a neighborhood $W_{x+y,1}$ of $(x+y)$ with $A(W_{x+y,1}) \subseteq V_{x+y}$. By continuity of addition in \mathbb{R}^n , we get neighborhoods $W_{x,1}, W_{y,1}$ of x and y , respectively, such that $W_{x,1} + W_{y,1} \subseteq W_{x+y,1}$. Thus

$$(2.1) \quad A(W_{x,1} + W_{y,1}) \subseteq A(W_{x+y,1}) \subseteq V_{x+y}.$$

On the other hand, by continuity of addition in \mathbb{R}^n , $Ax + Ay \in V_{xy}$ gives neighborhoods $V_{x,2}, V_{y,2}$ such that

$$(2.2) \quad Ax \in V_{x,2}, Ay \in V_{y,2} \text{ and } V_{x,2} + V_{y,2} \subseteq V_{xy}.$$

This implies there exist neighborhoods $W_{x,2}, W_{y,2}$ of x and y , respectively, with $A(W_{x,2}) \subseteq V_{x,2}$ and $A(W_{y,2}) \subseteq V_{y,2}$.

Define $W_x = W_{x,1} \cap W_{x,2}$, $W_y = W_{y,1} \cap W_{y,2}$. Then

$$(2.3) \quad \begin{aligned} A(W_x) &\subseteq A(W_{x,2}) \subseteq V_{x,2} = V_x \text{ (say)} \\ A(W_y) &\subseteq A(W_{y,2}) \subseteq V_{y,2} = V_y \text{ (say)} \\ A(W_x + W_y) &\subseteq A(W_{x,1} + W_{y,1}) \subseteq A(W_{x+y,1}) \subseteq V_{x+y} \end{aligned}$$

Choose $f_x, f_y \in \mathcal{S}$ such that $\text{Supp } f_x \subseteq W_x, \text{Supp } f_y \subseteq W_y$ and $f_x * f_y \not\equiv 0$. Let $g_x = Uf_x$ and $g_y = Uf_y$. Then $U(f_x * f_y) = g_x * g_y \not\equiv 0$.

We have

$$\begin{aligned}
 \text{Supp}(g_x * g_y) &= \text{Supp } U(f_x * f_y) \subseteq \text{Supp } Uf_x + \text{Supp } Uf_y \\
 &= A(\text{Supp } f_x) + A(\text{Supp } f_y) \subseteq AW_x + AW_y \\
 (2.4) \quad &\subseteq V_{x,2} + V_{y,2} \subseteq V_{xy} \quad (\text{by } 2.2)
 \end{aligned}$$

But $\text{Supp } (f_x * f_y) \subseteq \text{Supp } f_x + \text{Supp } f_y \subseteq W_x + W_y$. By (2.3), this gives

$$\begin{aligned}
 \text{Supp}(g_x * g_y) &= \text{Supp}(Uf_x * Uf_y) = \text{Supp } U(f_x * f_y) \\
 (2.5) \quad &= A(\text{Supp}(f_x * f_y)) \subseteq A(W_x + W_y) \subseteq V_{x+y}.
 \end{aligned}$$

From (2.4) and (2.5), we get

$$\text{Supp}(g_x * g_y) \subseteq V_{xy} \cap V_{x+y} = \emptyset.$$

This gives $g_x * g_y \equiv 0$, a contradiction. This proves the additivity of the map A .

Step 8. The map $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous additive bijection, and so also real linear. Hence it is given by an invertible matrix, which also we denote by A .

Step 9. 'Extension' of the map U to scalars.

Illustration of Step 9. For $f, g \in \mathcal{S}$, and $c(\neq 0) \in \mathbb{C}$, we have

$$U(cf)(x) U g(x) = U(cf g)(x) = U(f)(x) U(cg)(x), \quad x \in \mathbb{R}^n.$$

Let $h \in \mathcal{S}$ be such that $Uh(x) \neq 0$ for any $x \in \mathbb{R}^n$. Then we have

$$\begin{aligned}
 U(cf)(x) &= \frac{U(ch)(x)}{Uh(x)} Uf(x) \quad \text{for all } f \in \mathcal{S} \\
 &= m(c, x) Uf(x) \quad (\text{say}).
 \end{aligned}$$

Thus $U(cf)(x) = m(c, x) Uf(x)$, for all $x \in \mathbb{R}^n$. By definition, the function $m(\cdot, \cdot)$ is continuous in the second variable as a function of $x \in \mathbb{R}^n$.

Claim. The function $m(\cdot, \cdot)$ is independent of the second variable.

Proof of Claim. For $f, g \in \mathcal{S}$, $c \in \mathbb{C}$, and $x \in \mathbb{R}^n$, we have

$$\begin{aligned}
 U(cf * g)(x) &= U(f * cg)(x) \\
 (U(cf) * Ug)(x) &= (Uf * U(cg))(x) \\
 \int_{\mathbb{R}^n} m(c, x - y) Uf(x - y) Ug(y) dy &= \int_{\mathbb{R}^n} Uf(x - y) m(c, y) Ug(y) dy
 \end{aligned}$$

As the above equation holds good for all functions $f, g \in \mathcal{S}$, we have for all $F, G \in \mathcal{S}$,

$$\int_{\mathbb{R}^n} [m(c, x - y) - m(c, y)] F(x - y) G(y) dy = 0, \quad x \in \mathbb{R}^n.$$

Fix $x \in \mathbb{R}^n$. Let $G \in \mathcal{S}$ with $G = 1$ on $B(0, r)$, where $B(0, r)$ is the open ball in \mathbb{R}^n , centered at the origin and with radius r . Then for all functions $F \in \mathcal{C}_c^\infty$ with $\text{Supp } F \subseteq B(x, r)$, we have

$$\int_{B(0, r)} [m(c, x - y) - m(c, y)] F(x - y) dy = 0.$$

Thus $m(c, x - y) - m(c, y) = 0$ for all $y \in B(0, r)$.

by the continuity of the map $m(\cdot, \cdot)$ in the second variable. This gives in particular, $m(c, x) = m(c, 0)$. As x was arbitrary, the above argument gives that the function $m(c, x)$ is independent of the second variable $x \in \mathbb{R}^n$. We define

$$m(c) := m(c, 0).$$

Step 10. The map $m : \mathbb{C} \rightarrow \mathbb{C}$ is an additive and multiplicative bijection, which maps \mathbb{R} onto \mathbb{R} , and hence we have either $m(a) = a$ for all $a \in \mathbb{C}$, or $m(a) = \bar{a}$ for all $a \in \mathbb{C}$.

Proof of Step 10. Let $g \in \mathcal{S}$, $a, b \in \mathbb{C}$ with $g(x) \neq 0$ for any $x \in \mathbb{R}^n$. Then $Ug(y) \neq 0$ for any $y \in \mathbb{R}^n$.

Suppose $m(a) = m(b)$ for some $a, b \in \mathbb{C}$. Then

$$U(ag)(x) = m(a) Ug(x) = m(b) Ug(x) = U(bg)(x), \quad x \in \mathbb{R}^n.$$

Since U is a bijection, this gives $a = b$.

By hypothesis(1), we have

$$\begin{aligned} m(a + \bar{b}) Ug(x) &= U((a + \bar{b})g)(x) = U(ag + \bar{b}g)(x) \\ &= U(ag)(x) + \overline{U(\bar{b}g)}(x) = (m(a) + \overline{m(\bar{b})}) Ug(x). \end{aligned}$$

Since Ug is never zero, we get $m(a + \bar{b}) = m(a) + \overline{m(\bar{b})}$. In particular, $m(\bar{a}) = \overline{m(a)}$ for all $a \in \mathbb{C}$.

Now, hypothesis(2) gives

$$m(ab)Ug(x) = U(abg)(x) = m(a)U(bg)(x) = m(a)m(b)Ug(x).$$

Again, since Ug is nowhere vanishing, we get $m(ab) = m(a)m(b)$ for all $a, b \in \mathbb{C}$.

Step 11. For $f \in \mathcal{S}$, and $x_0 \in \mathbb{R}^n$, we have $Uf(Ax_0) = m(f(x_0))$.

Proof of Step 11. As before, choose $g \in \mathcal{S}$ such that $g(x) \neq 0$ for any $x \in \mathbb{R}^n$. Then $Ug(y) \neq 0$ for any $y \in \mathbb{R}^n$.

Define

$$h(x) := f(x_0) g(x) - f(x) g(x), \quad x \in \mathbb{R}^n.$$

Then $h \in \mathcal{S}$ and $h(x_0) = 0$. By Condition (\star) , we have $Uh(Ax_0) = 0$. This gives

$$\begin{aligned} 0 = Uh(Ax_0) &= U(f(x_0) \cdot g - f \cdot g)(Ax_0) \\ &= m(f(x_0)) Ug(Ax_0) - Uf(Ax_0) Ug(Ax_0). \end{aligned}$$

Since Ug is never zero, this gives $Uf(Ax_0) = m(f(x_0))$.

Since $B = A^{-1}$, using Step 10, we get that either $Uf(x_0) = f(Bx_0)$ or $Uf(x_0) = \overline{f(Bx_0)}$.

Thus we get that the map U is as claimed by our theorem. It remains to show that $|\det B| = 1$.

Step 12. The matrix B satisfies $|\det B| = 1$.

Proof of Step 12. We have

$$\begin{aligned} (f * g)(Bx) = U(f * g)(x) &= (Uf * Ug)(x) \\ &= \int_{\mathbb{R}^n} Uf(x - y) Ug(y) dy \\ &= \int_{\mathbb{R}^n} f(B(x - y)) g(By) dy \\ &= |\det B|^{-n} \int_{\mathbb{R}^n} f(Bx - y) g(y) dy \end{aligned}$$

Thus $|\det B| = 1$, proving our result. \square

References

- [1] S. Alesker, S. Artstein-Avidan and V. Milman, *A characterization of the Fourier transform and related topics*, C. R. Math Acad. Sci. Paris **346** (2008), 625-628.
- [2] S. Alesker, S. Artstein-Avidan and V. Milman, *A characterization of the Fourier transform and related topics*, Linear and Complex Analysis: Dedicated to V. P. Havin on the Occasion of his 75th Birthday, Advances in Mathematical Sciences, Amer. Math. Soc. Transl.(2) **226** (2009), 11-26.
- [3] P. Embrechts, *On a theorem of E. Lukacs*, Proc. Amer. Math. Soc. **68**(1978), 292-294. *Erratum in* Proc. Amer. Math. Soc. **75** (1979), 375.
- [4] C. E. Finol, *Linear transformations intertwining with group representations*, Notas de Matematica No. **63**, Universidad de Los Andes, Facultad de Ciencias, Departamento de Matematica, Merida-Venezuela, 1984.
- [5] P. Jaming, *A characterization of Fourier transforms*, Colloq. Math. **118** (2010), 569-580.
- [6] H. Kober, *On functional equations and bounded linear transformations*, Proc. London Math. Soc. (3) **14** (1964), 495-519.
- [7] E. Lukacs, *An essential property of the Fourier transforms of distribution functions*, Proc. Amer. Math. Soc. **3** (1952), 508-510.
- [8] R. Lakshmi Lavanya and S. Thangavelu, *Revisiting the Fourier transform on the Heisenberg group*, Publ. Mat. **58** (2014), No. 1, 47-63.
- [9] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*. **32**, Princeton Mathematical Series, Princeton University Press, Princeton, N.J., (1971).

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